Also solved by H. L. AGARD, T. M. BLAKSLEE, H. N. CARLETON, P. J. DA CUNHA, H. H. DOWNING, R. M. GINNINGS, C. E. MANGE, H. L. OLSON, LOUIS O'SHAUGHNESSY, ARTHUR PELLETIER, J. B. REYNOLDS, A. V. RICHARDSON, and J. H. WEAVER.

2838 [1920, 273–274].

"A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on to the other end. Suppose that the monkey begins to climb the rope, what will be the result?"

This problem was invented by Lewis Carroll in December, 1893 (S. D. Collingwood, *The Life and Letters of Lewis Carroll* (*Rev. C. L. Dodgson*), New York, 1899, pp. 317–318), and in his diary he remarked: "Got Professor Clifton's answer [R. B. Clifton, professor of physics at Oxford] to the 'Monkey and Weight Problem.' It is very curious, the different views taken by good mathematicians. Price [Bartholomew Price, professor of physics at Oxford] says that the weight goes *up* with increasing velocity; Clifton (and Harcourt [A. G. Vernon-Harcourt, professor of chemistry at Oxford]) that it goes *up*, at the same rate as the monkey; while Sampson [probably E. F. Sampson, lecturer, tutor and censor of Christ Church, Oxford] says that it goes *down*." Yet another solution by Rev. A. Brook is given on page 268 of *The Lewis Carroll Picture Book* . . . edited by S. D. Collingwood (London, 1899), namely, that "the weight remains stationary."

The problem has been recently discussed in *School Science and Mathematics*, volume 17, December, 1917, p. 821; volume 19, December, 1919, p. 815; and volume 20, February, 1920, pp. 172–173. The editors of the MONTHLY invite mathematical solutions of the problem.

The following solutions were contributed by request:

I. SOLUTION BY E. V. HUNTINGTON, Harvard University.

Case 1. If we neglect the weight of the pulley and rope, the solution follows immediately from the fundamental principle of mechanics, namely: the acceleration of a particle in any direction is proportional to the net force acting on the particle in that direction.

Here there are two particles to consider: (1) the monkey, and (2) the counterpoise.

The net upward force acting on the monkey is $T - W$, where W is the weight of the monkey, and T the tension in his part of the rope. The net upward force acting on the counterpoise is $T' - W'$, where W' is the weight of the counterpoise, and T' the tension in that part of the rope. But on the hypothesis of Case 1, the tension in the rope is the same at all points, so that $T' = T$; also, W' is known to be equal to W. Hence the net upward force acting on the monkey is the same as the net upward force acting on the counterweight, so that *the accelerations of the two bodies must be equal at every instant.*

Therefore, since the two bodies may be supposed to start from rest at the same level, their motions will be precisely parallel, no matter how fast or slow the monkey may climb, up or down, or how much he may allow the rope to slip through his hands.

Case 2. If we take into account the weight of the wheel (still neglecting the weight of the rope), we shall need to use also the equation of rotation.

Let w_0 be the weight of the wheel, r its radius, and k its radius of gyration. Then the equations of motion for the three bodies, W, W', and w_0, will be:

$$T - W = (W/g)dv/dt, \qquad T' - W' = (W'/g)dv'/dt,$$

and

$$Tr - T'r = (w_0/g)k^2 d\omega/dt,$$

where v and v' are the velocities of the monkey and the counterpoise, respectively, in the upward direction in space, and ω is the angular velocity of the wheel.

The geometric conditions of the problem tell us that as long as the rope does not become slack

$$v' = r\omega, \qquad \text{and} \qquad v' + v - u = 0,$$

where u is the relative velocity of the monkey up the rope.

From these five equations, we readily find:

$$\left(W + W' + \frac{w_0 k^2}{r^2}\right)\frac{dv'}{dt} = W\frac{du}{dt} + (W - W')g,$$

which gives the required acceleration, dv'/dt, of the counterpoise, when the relative acceleration, du/dt, of the monkey with respect to the rope is known.

Integrating twice, and putting $W = W'$, we find that the distance x risen by the counterpoise, when the monkey has climbed up a length s on the rope, is

$$x = s/\left(2 + \frac{w_0 k^2}{Wr^2}\right),$$

which reduces, as it should, to $x = s/2$ when $w_0 = 0$.

II. Solution by L. M. Hoskins, Stanford University.

If the problem is idealized by neglecting all friction and assuming the cord to be perfectly flexible and without mass, the solution is simple; the further assumption that the pulley is without mass simplifies it still further. The solution of this ideal problem does not furnish a satisfactory answer to the question, what will actually be the result if the monkey begins to climb the rope, for the result may depend in an important way upon the neglected factors. It is, however, useful to consider the simple idealized problem as a preliminary to a more general discussion taking account of friction and the mass of the rope.

(a) *Solution Neglecting Friction and Assuming the Cord and Pulley to be Without Mass.*—The initial condition is one in which both the monkey and the counterweight are at rest, each being in equilibrium under the action of two equal and opposite forces,—its own weight and the supporting pull exerted by the cord. In this initial condition the monkey is exerting a downward pull upon the cord equal to his weight; in beginning to climb he increases his pull in order to make the equal and opposite reacting pull exerted upon him greater than his weight, thus giving his center of mass an upward acceleration. Since the cord exerts upon the counterweight a pull equal to that exerted upon the monkey, the two bodies will have equal upward accelerations at every instant; and since both are initially at rest, they will always have equal velocities, and will move equal distances in any time. This agrees with the answer attributed to Clifton and Harcourt.

(b) *Effect of the Inertia of the Pulley.*—Let the same assumptions be made as in (a) except that the inertia of the pulley is not neglected. Let m denote the mass of the pulley, r its radius, mk^2 its moment of inertia about its axis of rotation, M the mass of the monkey and that of the counterweight, T and T_1 the upward pulls exerted by the cord on the counterweight and monkey respectively, a and a_1 their upward accelerations; the angular acceleration of the pulley will be a/r (assuming that the cord does not slip on the pulley). The following dynamical equations may be written:

For the monkey,

$$Ma_1 = T_1 - Mg;$$

for the counterweight,

$$Ma = T - Mg;$$

for the pulley,

$$mk^2 a/r = (T_1 - T)r.$$

From these equations, $T_1 - T = M(a_1 - a) = mk^2 a/r^2$; $a_1 = a(1 + mk^2/Mr^2)$. The accelerations of the two bodies are thus in a constant ratio, and the velocities acquired and distances described in any time, starting from rest, will be in the same constant ratio as the accelerations.

(c) *Effects of Friction and Weight of Cord.*—The weight of the cord causes the tension to vary between the pulley and each suspended body; let the values T, T_1 now refer to the points of tangency of the cord and pulley. Let F denote the difference between T and T_1 which will just maintain uniform rotation of the pulley; then F may be taken as a measure of the friction (including rigidity of the cord). Since F may vary with the velocity, let F_0 be its value for zero velocity (incipient motion). Initially the system is assumed to be at rest with $T = T_1 = Mg + W$, the last term representing the weight of cord below the pulley on each side; at the point of attachment of each of the suspended bodies the tension has the value Mg. In order to begin to climb (*i.e.*, to give to his center of mass an upward acceleration) the monkey has only to exert a pull greater than Mg; unless the pull exceeds Mg by more than F_0, $T_1 - T$ will not become greater than F_0, so that the pulley, cord and counterweight will remain at rest. The monkey will, however, move upward with increasing velocity so long as he maintains a pull which is greater than Mg by any amount whatever, and his velocity will continue undiminished even if the pull becomes equal to

Mg. If, therefore, the monkey wishes to climb the rope without disturbing the counterweight, he can do so if he is skillful enough to exert a pull greater than Mg but less than $Mg + F_0$. This conclusion is not affected by the inertia of the cord and pulley, nor by the weight of the cord.

It is conceivable, however, that the monkey might make such an exertion as to increase the tension by more than F_0; $T_1 - T$ would then become greater than F_0, so that the pulley would have an angular acceleration and the counterweight an upward acceleration. As soon as the counterweight has acquired any velocity it will continue to ascend with undiminished velocity so long as $T_1 - T$ is not less than F. The pull which the monkey would need to maintain to produce this result would be $Mg + F$ if the gravity forces remained balanced; actually the weight of the cord would immediately become unbalanced so that the requisite pull would be less than $Mg + F$ by an amount proportional to the length of cord that has passed the pulley (assuming that the free end of the rope does not reach the floor or other support). If, therefore, the object of the monkey is to raise the counterweight, he can accomplish it if he is able to exert a pull slightly greater than $Mg + F_0$ in order to start the motion. If he should relax his effort immediately, the system might be brought to rest by friction so that the effort would need to be repeated; but after the motion of the cord has proceeded far enough so that the unbalanced weight of the cord reaches the value F so as to overcome friction, the motion will continue without further effort on the part of the monkey. In this case if he wished to check the motion of the counterweight he would need to permit himself to have a downward acceleration; *i.e.*, he would have to produce the requisite decrease in the tension either by very active downward climbing or by letting go of the rope.

(*d*) *General Algebraic Solution.*—The dynamical basis of the foregoing discussion may be embodied in an equation. One method of procedure would be to write separate dynamical equations for the several bodies making up the system and then eliminate the internal forces, as was done above in (*b*). The final equation free from internal forces may, however, be written immediately by applying d'Alembert's principle to the entire system consisting of the monkey, counterweight, cord and pulley. Taking the axis of rotation of the pulley as axis of moments, let the sum of the moments of the mass-accelerations be equated to the sum of the moments of the external forces.

Let x denote the distance of the counterweight above its initial position at any instant, m' the total mass of the cord and ρ its mass per unit length; other notation being as above.

The moment of the weight of the rope is $2\rho gxr$, while the sum of the moments of all other gravity forces is zero, and the moment of the friction is $-Fr$. The only other external force is the normal axle pressure, the moment of which is zero.

The moments of the mass-accelerations are as follows: For the monkey, counterweight, and cord, $-Ma_1r$, Mar, and $m'ar$ respectively; for the pulley, mk^2a/r.

The equation is therefore

$$(M + m' + mk^2/r^2)ar - Ma_1r = 2\rho gxr - Fr,$$

or

$$a = \frac{2\rho gx + Ma_1 - F}{M + m' + mk^2/r^2}.$$

This equation holds when the counterweight is actually rising. Initially $x = 0$ and a will remain zero unless Ma_1 becomes greater than F_0; until this occurs F is to be regarded as representing the actual friction (less than the limiting value F_0) and is equal to Ma_1, so that the equation gives $a = 0$ as it should. After the counterweight has moved from its initial position so that x is no longer 0, a will have a positive value so long as $Ma_1 > F - 2\rho gx$. If $2\rho gx$ becomes $> F$, a will remain positive even if a_1 becomes 0. The equation shows, however, that a can be made negative by giving a_1 a sufficiently great negative value. The conclusions stated under (*c*) are in fact all implied by the above general equation.

The equation also covers the ideal cases (*a*) and (*b*); in the latter it is assumed that ρ, m' and F are zero, while in (*a*) m is also assumed to be zero.

Since $a = d^2x/dt^2$, the general equation is a differential equation of the second order, a_1 being a function of t which depends upon the activity of the monkey, while F is an unknown function of the velocity. If F is treated as constant the equation becomes linear, and may be solved if a_1 is constant or a known function of t.

Thus if, starting from the original balanced condition, a_1 keeps a constant value a_1' for a

certain time, the solution for this part of the motion is

$$x = \frac{Ma_1' - F}{2\rho g}\left[\cosh t\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right].$$

If, at the instant $t = t_1$, a_1 changes to the constant value a_1'', the solution for the ensuing motion is

$$x = \frac{Ma_1' - F}{2\rho g}\left[\cosh t\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right] + M\frac{(a_1'' - a_1')}{2\rho g}\left[\cosh (t - t_1)\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right].$$

Professor Huntington's case 1 was also solved by C. C. WYLIE.

2863 [1920, 482]. Proposed by A. A. BENNETT, University of Texas.

From their generation as roulette curves, show that the two hypocycloids of five cusps drawn with common vertices, are such that each is the envelope of a chord of constant length suitably placed upon the other.

Show that for any odd prime p, the $(p - 1)/2$ distinct p-cusped hypocycloids with common vertices may be arranged in cycles, so that each is the envelope of a chord of constant length taken upon the succeeding curve of the cycle.

A solution of this problem appears on pages 371–373 of this issue of the MONTHLY.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

Miss GERTRUDE I. MCCAIN, of Oxford College, has been made professor of mathematics at Westminster College, New Wilmington, Pa.

Associate Professor J. V. MCKELVEY decided to remain at Iowa State College (compare *1921*, 285).

We are requested to state that our note regarding Associate Professor W. A. WILSON's promotion (*1921*, 332) is not in accordance with fact.

At the University of Michigan, Mr. J. P. BALLANTINE, of Pennsylvania State College, and Mr. W. M. COATES, of the University of Virginia, have been appointed instructors of mathematics.

Mr. J. C. FUNK, of Tamalpais Polytechnic High School, Mill Valley, Cal., has been elected to the headship of the department of mathematics in the Santa Maria high school and junior college.

Associate Professor EMMA L. KONANTZ, who has spent two years on leave of absence teaching in Peking University, has returned to her position at Ohio Wesleyan University.

Mr. H. K. CUMMINGS, instructor of physics at the Worcester Polytechnic Institute, and experimental physicist in the research laboratories of the Acheson Graphite Company, has been appointed instructor of mathematics at Brown University.

Mr. F. W. WINTERS, of Mount Allison University, but recently a student instructor at Yale University, has been appointed assistant professor of mathematics at Dalhousie College, Halifax, Nova Scotia.